Nambu-Goto string action with Gauss-Bonnet term[*]

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Abstract

We examine the relativistic Nambu–Goto model with Gauss–Bonnet boundary term added to the action integral. The dynamical system is analysed using an invariant representation of string degrees of freedom by complex Liouville fields. The solutions of classical equations of motion for open strings are described.

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Relativistic string theories are often employed to study the dynamics of one–dimensional objects which may arise and play a crucial role in some more fundamental microscopic models. In particular, open strings appear there as an approximate description of some realistic linear objects that spread between some points or sources (vortices with monopoles at their ends, flux tubes between quarks etc.). Even though the problem of quantizing relativistic strings is still cumbrous, it is widely believed that some kind of useful effective theoretical string description should exist for all those various phenomena that reveal an appealing 'stringy' character (e.g. dynamics of linear topological defects in theories with spontaneous symmetry breaking, fluctuactions of thin color flux tubes in the confining phase of non–Abelian gauge theories). The simplest string theory, Nambu–Goto model, is usually useful to reproduce some most important qualitative results, but it fails as we carry the program forward. There are many directions in which one can modify the Nambu–Goto model. As far as open strings are concerned, one might be tempted to embody new features to the system by considering only interactions between a string and string ends.

In previous papers [1, 2] a suitable formalism was constructed that described classical dynamics of open strings with 'interactions at ends'. In light of these works, we consider here a particular example of extended Nambu–Goto model, when Gauss–Bonnet term is added to the action. This model has been examined earlier in other works [3–7]. Zheltukhin [3] considered the system embedded in three–dimensional spacetime and found the correspondence between classical string solutions and solutions of the real Liouville equation with Dirichlet–type boundary conditions. However, only static solutions have been found and discussed. Barbashov and collaborators [4–6] worked out the classical system of equations eliminating gauge freedom (choice of parametrization) and derived some particular type of solutions. In the presented paper, we re-examine the model and find all types of classical solutions. The most interesting result is that only planar string configurations can be extrema of the classical action.

The classical string model discussed by us is defined by the following action integral:

$$S = -\gamma A - \frac{\alpha}{2} S_{GB} \; , \tag{1}$$

where γ stands for string tension, A denotes world sheet area, α is a dimensionless parameter. Parameters γ and α are arbitrary positive real constants.

1

S_{GB} is a pseudoeuclidean version of Gauss–Bonnet term, being the total integral of internal world sheet curvature. The string action (1) shows no difference with the two–dimensional gravity model, a cosmological (Nambu–Goto) term is supplemented by Einstein (Gauss–Bonnet) term. As Einstein Lagrangian is a total derivative in two dimensions, this action term acts only at boundaries, modifying the sector of open strings.

Classical string equations of motion derived from (1) are Nambu–Goto equations together with some 'edge conditions', being in fact dynamical equations involving third order time derivatives. As usual in the case of minimal surfaces, their general solution can be represented as the combination of left– and right–moving parts,

$$X_\mu(\tau, \sigma) = X_{L\mu}(\tau + \sigma) + X_{R\mu}(\tau - \sigma), \tag{2}$$

where the world sheet parametrization has been defined by the following conditions [8]:

$$\begin{aligned} (\dot{X} \pm X')^2 &= 0, \\ (\ddot{X} \pm \dot{X}')^2 &= -q^2. \end{aligned} \tag{3}$$

Bulk equations of motion get linearized due to the orthonormal gauge. As this gauge still allows for conformal changes of parametrization, so that this residual gauge freedom has been saturated by the latter pair of equations in (3). The parameter q has dimension of mass and can take any positive real value. This is an integral of motion, which fixes the scale for considered solutions.

Up to transformations of Poincare group, minimal surfaces X_μ parametrized according to (3) correspond to solutions Φ of the complex Liouville equation [1, 2]:

$$\ddot{\Phi} - \Phi'' = 2q^2 e^\Phi. \tag{4}$$

This correspondence can be presented in the following way:

$$\begin{aligned} e^\Phi &= -\frac{4}{q^2} \frac{f'_L(\tau + \sigma) f'_R(\tau - \sigma)}{[f_L(\tau + \sigma) - f_R(\tau - \sigma)]^2}, \\ \dot{X}^\mu_{L,R} &= \frac{q}{4|f'_{L,R}|}(1 + |f_{L,R}|^2, \ 2\mathrm{Re}\, f_{L,R}, \ 2\mathrm{Im}\, f_{L,R}, \ 1 - |f_{L,R}|^2), \end{aligned} \tag{5}$$

where f_L and f_R are arbitrary complex functions. Note that any simultaneous modular transformation of f_L and f_R induces Lorentz transformation of X_μ

while Liouville field Φ remains invariant. For later discussions, it is helpful to invert relations (5):

$$S(f_{L,R}) = \partial_\pm^2 \Phi - \frac{1}{2}(\partial_\pm \Phi)^2 \ ,$$

$$f_{L,R} = \frac{\dot{X}_{L,R}^1 + i\dot{X}_{L,R}^2}{\dot{X}_{L,R}^0 + \dot{X}_{L,R}^3} \ , \tag{6}$$

where $\partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1)$, and $S(f)$ stands for Schwartzian derivative:

$$S(f) = \frac{f'''}{f'} - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 \ .$$

All classical solutions of the Nambu–Goto string model with Gauss–Bonnet boundary term added, represented as above (5), are subject to the following conditions imposed at string ends [2, 7]:

$$e^\Phi = -\frac{1}{q}\sqrt{\frac{\gamma}{\alpha}} \equiv -\frac{C^2}{q^2} \ , \tag{7}$$

$$\mathrm{Im}\,\Phi' = 0 \quad \text{for} \quad \sigma = 0,\ \pi \ . \tag{8}$$

It is helpful for later analysis to discuss possible singularities of Liouville fields. Since $\mathrm{Im}\Phi$ is some angle variable, the singularities refer only to $\mathrm{Re}\Phi$. Singular points can be taken into account only if $\mathrm{Re}\Phi > 0$ in their vicinities [2]. It implies that at any world sheet point we have [2]:

a) derivatives f_L' and f_R' cannot take zero values;

b) ratios f_L^2/f_L' and f_R^2/f_R' are finite $\qquad$ (9)

The singularities of the Liouville field with $\mathrm{Re}\Phi > 0$ correspond to singularities of the induced metric: $g = 0$.

Let us now start to study boundary conditions (7, 8). First, we note that Liouville field Φ and its first and second derivatives are real at boundary points $\sigma = 0, \pi$. Thus, it follows from relations (6) that Schwartzian derivatives of f_L and f_R are real for any values of their arguments. Henceforth, one can always find a modular transformation that turns either f_L or f_R into unimodular functions. Let us specify the following choice:

$$f_L' = e^{iF_L} \ , \quad f_R' = \frac{ae^{iF_R} + b}{ce^{iF_R} + d} \ , \tag{10}$$

3

where F_L and F_R are some real functions, and complex coefficients of the modular transformation satisfy: $ad - bc = 1$. Furthermore, taking into account imaginary part of Eq.(7) and Eq.(8) one can easily obtain that

$$a = \pm\bar{d} \ , \quad b = \pm\bar{c} \ . \tag{11}$$

This implies that the function f_R must be unimodular as well. After some simple redefinition, we have now

$$f_L = e^{iF_L} \ \text{ and } \ f_R = e^{iF_R} \tag{12}$$

and

$$e^{\Phi} \ = \ -\frac{1}{q^2} \frac{F'_L F'_R}{\sin^2 \frac{F_L - F_R}{2}} \ , \tag{13}$$

$$\dot{X}^{\mu}_{L,R} \ = \ \frac{q}{2|F'_{L,R}|}(1, \ \cos F_{L,R}, \ \sin F_{L,R}, \ 0) \ . \tag{14}$$

As a consequence of the boundary conditions, the imaginary part of Liouville field Φ is constant and equal π at any point of the world sheet. Thus, a surprising conclusion has been drawn from the above analysis of boundary conditions (7,8) – any particular classical solution of the equations of motion describes a string which time evolution entirely takes place in some fixed plane. In other words, dynamics of extrinsic geometry of the world sheet (described by $\mathrm{Im}\Phi$) is frozen. The world sheet is immersed in some three–dimensional plain hyper–surface. A unit vector normal to this hyper–surface is an integral of motion and can be used to specify different solutions.

Next, from boundary conditions (7) and from requirements (9) we obtain that

$$F'_L F'_R > 0 \ , \tag{15}$$

at any point of the world sheet. It means that both real functions F_L and F_R are either increasing or decreasing.

If we have already shown that only planar solutions can exist, let us attempt to solve boundary problem (7,8) for the complex Liouville equation completely. We will show that all solutions can be divided into three groups:

A. periodic solutions:

$$f_L(\tau) = f_L(\tau + 2\pi) \text{ and } f_R(\tau) = f_R(\tau + 2\pi) \ ;$$

4

B. 'furled' strings:

$$f_L(\tau) = \lambda\tau \text{ and } f_R(\tau) = \lambda(\tau + \pi);$$

C. non–periodic solutions:

$$f_L(\tau) \neq f_L(\tau + 2\pi) \text{ and } f'_L(\tau) \neq f'_L(\tau + 2\pi) ,$$

or:

$$f_R(\tau) \neq f_R(\tau + 2\pi) \text{ and } f'_R(\tau) \neq f'_R(\tau + 2\pi) .$$

It is important to note that we consider only the period 2π, where π is distinguished here as it is the length of σ–interval. Therefore, periodic solutions with other periods, if they exist, do not belong to the class A.

CASE A:

In the case A, solutions have the following general form:

$$F'_L = \frac{1}{2}\left[h' \pm \sqrt{h'^2 + 4C^2 \sin^2(h/2)}\right] ,$$
$$F'_R = \frac{1}{2}\left[-h' \pm \sqrt{h'^2 + 4C^2 \sin^2(h/2)}\right] , \tag{16}$$

where constant C^2 has been defined in (7) and h is some periodic (with period 2π) function. Its arbitrariness is restricted by the following constraints:

$$h(\tau + 2\pi) = h(\tau) ,$$
$$h(\tau) \neq 2k\pi \text{ for any } \tau \text{ and integer } k ,$$
$$\int_0^{2\pi} d\tau \sqrt{h'^2 + 4C^2 \sin^2(h/2)} = 2N , \tag{17}$$

where N is some natural number. The conditions (17) follow from periodicity (case A) and non–singularity (15) requirements. One can concince oneself that there exist infinitely many functions h obeying the above requirements, so that there are infinitely many examples of periodic solutions.

As h in (16) we can take an arbitrary periodic function which is a mapping on any finite interval that does not include points $2k\pi$. For any given N, we can always match a contant of motion C according to (17). Finally, periodicity requirements stated in type A are assured if we adjust integration constants while solving (16) in such a way that

$$F_L - F_R = h . \tag{18}$$

5

CASE B:

If some non-periodic function either f_L or f_R has a periodic first derivative, then one can derive easily from boundary conditions (7,8) that $f_L'' = f_R'' = 0$. Therefore, we obtain the solution B, which represent a folded straight string propagating with some longitudinal velocity. The constant q is given by

$$q = \frac{4}{\pi^2}\sqrt{\frac{\alpha}{\gamma}} \ .$$

CASE C:

In the case C, boundary conditions (7) enable us to find the explicite relation between left- and right-movers:

$$F_R(\tau) = F_L(\tau + 2\pi) +$$
$$\text{arc ctg}\left[\pm \sin^{-1}(\Delta F_L) - \text{ctg}(\Delta F_L) \right] , \qquad (19)$$

where

$$\Delta F_L \equiv \frac{F_L(\tau + 2\pi) - F_L(\tau)}{2} \ .$$

The function F_L itself is subject to the following equation:

$$\frac{F_L''(\tau + 2\pi)}{F_L'(\tau + 2\pi)} - \frac{F_L''(\tau)}{F_L'(\tau)} - [F_L'(\tau + 2\pi) + F_L'(\tau)]\text{ctg}(\Delta F_L)$$
$$= \pm\frac{C^2 - 2G_L^2}{G_L} , \qquad (20)$$

where

$$G_L \equiv \frac{\sqrt{F_L'(\tau + 2\pi)F_L'(\tau)}}{\sin(\Delta F_L)} \ .$$

First, let us discuss how many solutions of type C there exist. Let us freely define the function F_L in the interval $[0, 2\pi]$ ensuring only that the values of the function and its first and second derivatives at points 0 and 2π satisfy (20). Now, we can extend the definition of F_L to the interval $[2\pi, 4\pi]$ (or to the interval $[-2\pi, 0]$) by solving differential equation (20) for $F_L(\tau)$ with given retarded $F_L(\tau - 2\pi)$ (or advanced $F_L(\tau + 2\pi)$) values taken from the

6

interval $[0, 2\pi]$. Following this procedure, starting from arbitrary $\mathcal{C}$ function F_L defined in the interval $[0, 2\pi]$ which values at end points satisfy algebraic equations (20), we can construct a $\mathcal{C}^2$ solution to differential equation (20). The construction described here (see also [9, 10]) provides also a basis of an algorithm to generate numerical solutions.

The above construction can be applied unless the function e^{iF_L} is periodic. If it happens, we can rearrange the construction using the other function F_R. However, in case either F_L or F_R is periodic we can use a simpler method to derive solutions. We are looking for the function $h = F_L - F_R$ which can be derived either from the equation

$$(F_L')^2 - F_L' h' = C^2 \sin^2 \frac{h}{2} \ , \tag{21}$$

with a given periodic function e^{iF_L}, or from the equation

$$(F_R')^2 + F_R' h' = C^2 \sin^2 \frac{h}{2} \ , \tag{22}$$

with a given periodic function e^{iF_R}.

Let us give some examples of solutions. The rotating rigid rod solution [7] is given by

$$F_L^{(0)}(\tau) = \pm\lambda(\tau - \frac{\pi}{2}) \ , \quad F_R^{(0)}(\tau) = \pm\lambda(\tau + \frac{\pi}{2}) + \pi \ , \tag{23}$$

where frequency λ satisfies

$$\lambda^2 = q\sqrt{\frac{\gamma}{\alpha}} \cos^2 \frac{\lambda\pi}{2} \ .$$

The plus and minus signs correspond to clockwise and anti–clockwise revolutions respectively.

The rotating rigid rod solution is a static (solitonic) solution of the Liouville equation (4). We perturbed this solution and solving numerically the equation (20) (according to described algorithm) we obtained a solution depicted in Fig.1 (F_L' is represented here). This exact numerical solution can be compared with the approximate general solution derived in [7] for perturbations around the static solution:

$$F_{L,R} = \frac{\pi}{2} \mp \frac{\pi}{2} \pm \sum_{n/1}^{\infty} D_n \sin\left[\omega_n(\tau \mp \frac{\pi}{2}) + \varphi_n \pm \frac{n\pi}{2}\right], \tag{24}$$

7

where plus and minus signs correspond to left– and right–movers respectively, and D_n, φ_n are arbitrary real constants. The eigenfrequencies ω_n can be calculated from the transcendental equation:

$$\omega_n \text{tg}\frac{\pi(\omega_n + n)}{2} = \lambda\text{tg}\frac{\pi\lambda}{2} \ .$$

An approximate solution obtained from the above series, which corresponds to the same initial data as the numerical one, is almost exactly the same as that plotted in Fig.1.

The special case associated with the equation (21) can be illustrated with the following exact solution:

$$\begin{aligned} F_L &= N\tau \ , \\ F_R &= N\tau + 2\text{arc tg}(e^{N\tau}) - \frac{3\pi}{2} \ , \end{aligned} \tag{25}$$

where N is integer and $C^2 = 2N^2$.

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Figure Caption

Fig.1. A numerical solution to the equation (20). The first derivative of F_L is plotted.

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